



ECOLE  
**POLYTECHNIQUE**  
DE BRUXELLES

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**MATH-H401 Project**  
**Numerical Methods for PDEs**

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# 1. Introduction

This report as been written with the aim to report the result of the numerical resolution of the following boundary value problem :

$$\begin{cases} -\Delta u = 0 & \text{on } \Omega \subset \mathbb{R}^2, \\ \text{boundary conditions} & \text{on } \partial\Omega, \end{cases} \quad (1)$$

The boundary conditions implemented in the code are :

$$u = \sin(\sqrt{x^2 + y^2}) \quad \text{on } \partial\Omega. \quad (2)$$

and

$$u = \exp(\sqrt{x^2 + y^2}) \quad \text{on } \partial\Omega. \quad (3)$$

The geometry of the problem used to compare the different results is :

$$\Omega = ([0, 6] \times [0, 10]) \setminus ([2, 4] \times [2, 5]) \quad (4)$$

That look like :

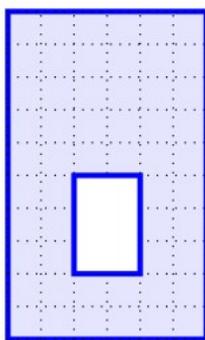


Figure 1: Domain

The code can adapt to different geometries but only this one is used to keep a good comparison between the results of the different algorithms

## 2. Discretization

In order to solve this problem, we need to discretize our domain, and use the finite difference method to find the value of each discrete point in function of its surrounding (North, South, Est, West):

$$\frac{4u(P) - u(S) - u(W) - u(E) - u(N)}{h^2} = O(h^2) \quad (5)$$

with  $h$  being the distance between 2 points. This lead to a linear system to solve:

$$Au = b \quad (6)$$

where  $b$  hold the boundary conditions and  $u$  is the solution of the problem on each discrete point.

The level of discretization will be characterize by the number  $m$ , representing the number of points on the smallest side of the domain , boundary included. The problem will be solve on this range of  $m$  values : 1537, 769, 385, 193, 97, 49, 25, 13, 7.

## 3. Two-Grid Method

The Two-Grid method is the first algorithm used to approximate our solution. It is the addition of stationnary iteration algorithms (smoothing) and the approximation of the solution on a coarser level of discretization. The program exec0ute 4 forward Gauss-Seidel iterations on the main level ( $h$ ), once the UMFPACK method on the coarser level ( $2h$ ) and 4 backward Gauss-Seidel Iteration on the main level.

The UMFPACK function killed the program every time we try to solve the problem for higher  $m$  than 385. The maximum  $M$  was therefore 769.

We can notice 2 things : the residue stabilizes after 9 iterations of the Two Grid algorithm, and for some unknown reasons, the precision of the result decrease with higer  $m$ .

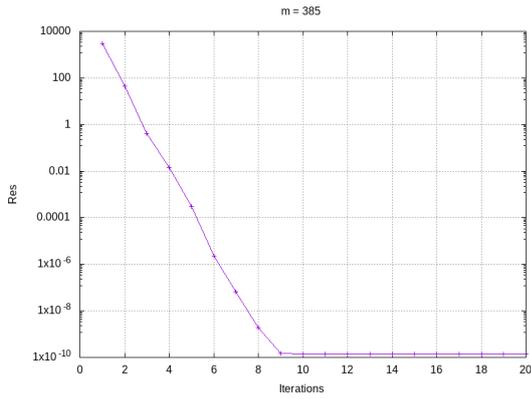


Figure 2: TG Method for  $m = 385$

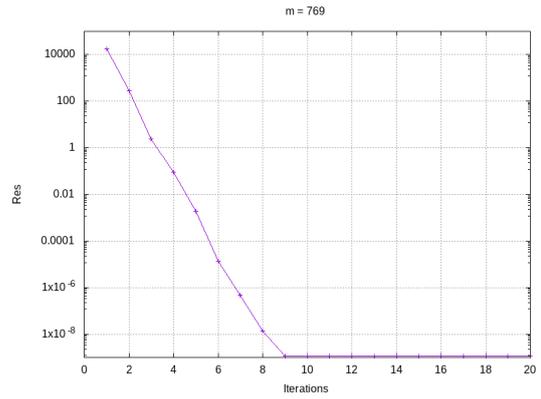


Figure 3: TG Method for  $m = 769$

## 4. Multigrid Method

The Multigrid method is a recursive implementation of the Two-Grid method. The basic version of this method is the V cycle. The method will coarse the problem until the bottom level (here corresponding to  $m = 7$ ). In this project the W and F cycle have also been implemented.

Let's first look at the residue for  $m = 769$  for the 3 different cycle:

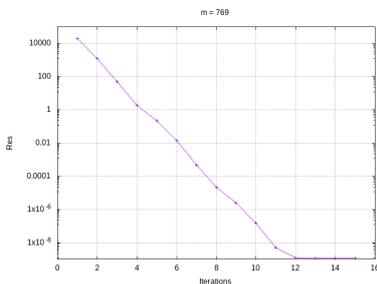


Figure 4: MG method for  $m = 769$  V cycle

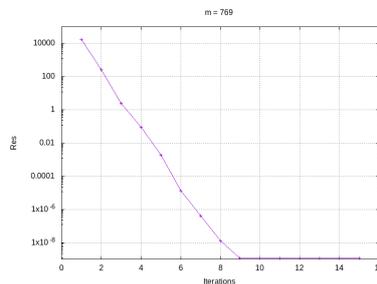


Figure 5: MG method for  $m = 769$  W cycle

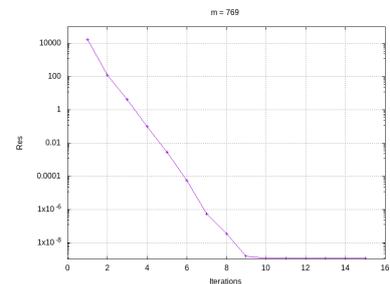


Figure 6: MG method for  $m = 769$  F cycle

Once again we directly notice that the norm of the residue isn't reaching the machine precision. For the V cycle the residue stabilizes around 12 iterations, and for the W and F cycle it is more around 9-10 iterations, it seems logic since they are more demanding in resource, and are visiting more the coarser levels

Now for  $m = 1537$ :

The conclusion are the same as for the previous  $m$ , we can notice each time the problem discretization level double the precision is divided by 10. This fact might help us for further improvements of the project.

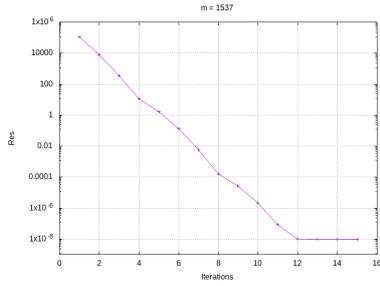


Figure 7: MG method for  $m = 1537$  V cycle

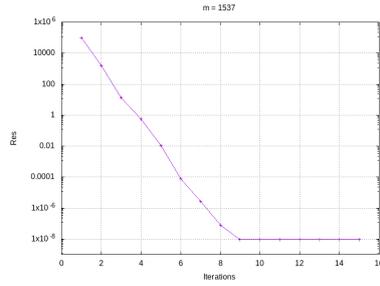


Figure 8: MG method for  $m = 1537$  W cycle

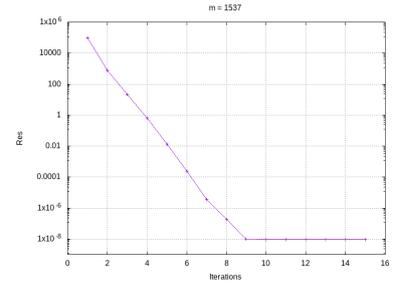


Figure 9: MG method for  $m = 1537$  F cycle

## 5. Multigrid preconditioning

The last method used in this project mixes Multigrid and Conjugate Gradient method. The CG is an iterative algorithm designed for symmetric positive-definite matrices, and it works by minimizing the energy norm of the residual. Furthermore, the Multigrid method will here serve as a preconditionner of our matrix  $A$ , and will be use to approximate with 2 iterations the system ( in V cycle):

$$Ax = r \tag{7}$$

Let's loo at the result for  $m = 769$  and  $m = 1537$ :

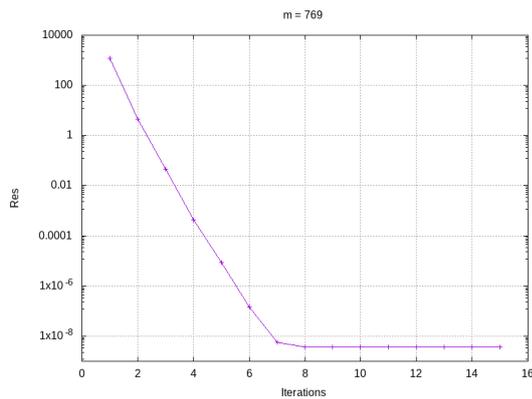


Figure 10: CG method for  $m = 769$

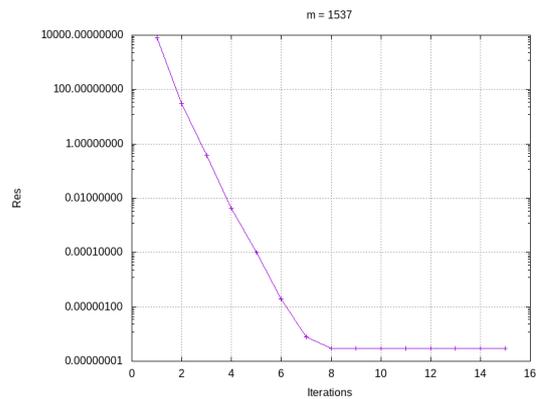


Figure 11: CG method for  $m = 1537$

We can see that the method stabilize after 8 iterations

## 6. Conclusion

The results are showing signs of convergence but not at good values of the norm of the residue. The problem is recurrent over all the method which leads to think that the issue is not related to the implementation of the method in itself but maybe somewhere else in the code.